SUCCESSIVE RADII AND MINKOWSKI ADDITION

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Abstract. In this paper we study the behavior of the so called successive inner and outer radii with respect to the Minkowski addition of convex bodies, generalizing the well-known cases of the diameter, minimal width, circumradius and inradius. We get all possible upper and lower bounds for the radii of the sum of two convex bodies in terms of the sum of the corresponding radii.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively, and denote by e_i the i-th canonical unit vector. Let B_n be the n-dimensional unit ball.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $L\in\mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement and for the sake of brevity we write $B_{i,L}=B_n\cap L$. For $K\in\mathcal{K}^n$ and $L\in\mathcal{L}_i^n$, the orthogonal projection of K onto L is denoted by $K|L$. With $\mathrm{lin}\{u_1,\dots,u_m\}$ we represent the linear hull of the vectors $u_1,\dots,u_m$ and with $[u_1,u_2]$ the line segment with end-points u_1,u_2. Finally, for $S\subset\mathbb{R}^n$ we denote by $\mathrm{conv}\,S$ the convex hull of S and by $\mathrm{bd}\,S$ its boundary. Moreover, we write $\mathrm{relbd}\,S$ to denote the relative boundary of S, i.e., the boundary of S relative to its affine hull $\mathrm{aff}\,S$.

The diameter and the minimal width of a convex body K (respectively, the maximum and the minimum distance between two parallel support hyperplanes of K), the circumradius and the inradius of K (the radius of, respectively, the smallest ball containing K and one of the greatest balls contained in K) are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [4, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine space A, then we write $f(K;A)$ to stress that f has to be evaluated with respect to the space A. The successive outer radii R_i and inner radii r_i are defined in the following way.

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Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let*

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \text{and} \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\bigl(K \cap (x+L); x+L\bigr).$$

It is clear that the outer radii are increasing in i, whereas the inner radii are decreasing in i. Observe that $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with i-dimensional spherical cross section containing K, whereas $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K, and we obviously have

$$\mathrm{R}_n(K) = \mathrm{R}(K), \quad \mathrm{R}_1(K) = \frac{\omega(K)}{2}, \quad \mathrm{r}_n(K) = \mathrm{r}(K), \quad \mathrm{r}_1(K) = \frac{\mathrm{D}(K)}{2}.$$

The first systematic study of the successive radii was developed in [2]. For more information on these radii, their size for special bodies and their relation with other measures, as well as computational aspects of the radii we refer to [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]. We mention, in particular, the following inequalities: for $i \in \{1, \dots, n\}$ and any convex body K,

$$1 \leq \frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i+1. \tag{1}$$

For the lower bound, which is best possible, we refer to [2, Lemma 2.1]. To determine the optimal upper bound is still an open problem, even in the 0-symmetric case. The bound presented above is given in [14] (see also [13]).

Here, however, we are mainly interested in the relations of these radii to the Minkowski sum (i.e., vectorial addition) of convex bodies. The behavior of the diameter, minimal width, circumradius and inradius with respect to the Minkowski sum is well known (see e.g. [15, p. 42]), namely,

$$\begin{aligned} &\mathrm{D}(K+K') \leq \mathrm{D}(K) + \mathrm{D}(K'), && \omega(K+K') \geq \omega(K) + \omega(K'), \\ &\mathrm{R}(K+K') \leq \mathrm{R}(K) + \mathrm{R}(K'), && \mathrm{r}(K+K') \geq \mathrm{r}(K) + \mathrm{r}(K'), \end{aligned} \tag{2}$$

which can be translated as inequalities for r_1, R_1, R_n, and r_n, respectively. Hence the question arises to study the relation between Minkowski addition and the remaining successive inner and outer radii. Regarding outer radii we prove the following theorem.

Theorem 1.1. *Let $K, K' \in \mathcal{K}^n$. Then*

$$\begin{aligned} \mathrm{R}_1(K+K') &\geq \mathrm{R}_1(K) + \mathrm{R}_1(K'), \\ \sqrt{2}\mathrm{R}_i(K+K') &\geq \mathrm{R}_i(K) + \mathrm{R}_i(K'), \quad i = 2, \dots, n. \end{aligned} \tag{3}$$

All inequalities are best possible.

Moreover, $\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$ (cf. (2)) and there exists no constant $c > 0$ such that $c\,\mathrm{R}_i(K+K') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$ for $i = 1, \dots, n-1$ (see Remark 3.1).

In the case of the successive inner radii, the result is the following.

Theorem 1.2. *Let $K, K' \in \mathcal{K}^n$. Then*

$$
\begin{aligned}
\sqrt{2}\mathrm{r}_i(K+K') &\geq \mathrm{r}_i(K) + \mathrm{r}_i(K'), \quad i = 1, \dots, n-1, \\
\mathrm{r}_n(K+K') &\geq \mathrm{r}_n(K) + \mathrm{r}_n(K').
\end{aligned} \tag{4}
$$

All inequalities are best possible.

Moreover, $\mathrm{r}_1(K+K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$ (cf. (2)) and there exists no constant $c > 0$ such that $c\,\mathrm{r}_i(K+K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$ for $i = 2, \dots, n$ (see Remark 3.2).

The paper is organized as follows. In Section 2 we give preliminary lemmas which are needed for the proof of Theorem 1.2. Then, in Section 3 we present the proofs of the main theorems, as well as some consequences and remarks. Finally, Section 4 is devoted to study particular cases for which the bounds in Theorems 1.1 and 1.2 can be improved.

2. Some preliminary results

We state here some preliminary results in Linear Algebra which will be needed in the proof of the main theorems.

Lemma 2.1. *For $1 \leq i \leq n/2$, let $L, L' \in \mathcal{L}^n_i$ be such that $L \cap L' = \{0\}$. Then there exist orthonormal bases $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ of L and L' respectively, such that the 2-dimensional subspaces $\mathrm{lin}\{u_1, v_1\}, \dots, \mathrm{lin}\{u_i, v_i\}$ are pairwise orthogonal.*

Proof. Throughout the proof we will always work with $L + L'$ as the main vector space instead of $\mathbb{R}^n$ when considering subspaces, orthogonal complements, projections... Moreover we will identify $L + L' \equiv \mathbb{R}^{2i}$ for the sake of brevity. We distinguish two cases.

Case (i): First we suppose that $L \cap L'^{\perp} = \{0\}$. Then, denoting by π' the orthogonal projection onto L', it clearly holds that $\pi'(L) = L'$. We assume, without loss of generality, that $L' = \mathrm{lin}\{\mathrm{e}_{i+1}, \dots, \mathrm{e}_{2i}\}$, and let $w_j \in L$ be such that $\pi'(w_j) = \mathrm{e}_{i+j}$, $j = 1, \dots, i$. Let $W = (w_1 \cdots w_i) \in \mathbb{R}^{2i \times i}$ be the $(2i \times i)$-matrix with column vectors w_j, which takes the form

$$W = \begin{pmatrix} M \\ \mathrm{I}_i \end{pmatrix}, \quad M \in \mathbb{R}^{i \times i}.$$

Here I_i denotes the $(i \times i)$-identity matrix. Then the singular value decomposition of a real matrix (see e.g. [12, p. 80]) ensures the existence of orthogonal matrices $U, V \in \mathbb{R}^{i \times i}$ and a diagonal matrix $D = \mathrm{diag}\{d_1, \dots, d_i\}$ such that $U^{\intercal} M V = D$. We write $U = (u'_1 \cdots u'_i)$ and $V = (v'_1 \cdots v'_i)$, with $u'_j = (u'_{j1}, \dots, u'_{ji})^{\intercal}$ and $v'_j = (v'_{j1}, \dots, v'_{ji})^{\intercal}$. Notice that, on the one hand, $WV = \left(\sum_{k=1}^{i} v'_{1k} w_k \cdots \sum_{k=1}^{i} v'_{ik} w_k\right)$, i.e., the column vectors of WV are linear combinations of $\{w_1, \dots, w_i\}$. So they lie in L. On the other hand,

$$WV = \begin{pmatrix} M \\ \mathrm{I}_i \end{pmatrix} V = \begin{pmatrix} MV \\ V \end{pmatrix} = \begin{pmatrix} UD \\ V \end{pmatrix} = \begin{pmatrix} d_1 u'_1 & \cdots & d_i u'_i \\ v'_1 & \cdots & v'_i \end{pmatrix}.$$

Therefore, the column vectors $(d_j u'_j, v'_j)^\intercal \in L$ for all $j = 1, \dots, i$. Notice that $d_j \neq 0$, $j = 1, \dots, i$, otherwise we would get $(0, v'_j)^\intercal \in L \cap L' = \{0\}$, a contradiction. Moreover, $\bigl\{(d_j u'_j, v'_j)^\intercal : j = 1, \dots, i\bigr\}$ are non-zero pairwise orthogonal vectors, since

$$\bigl\langle (d_j u'_j, v'_j)^\intercal, (d_k u'_k, v'_k)^\intercal \bigr\rangle = \bigl\langle d_j u'_j, d_k u'_k \bigr\rangle + \bigl\langle v'_j, v'_k \bigr\rangle = 0$$

for all $j \neq k$, $j, k \in \{1, \dots, i\}$, because U, V are orthogonal matrices. Then, we define the vectors

$$u_j = \frac{1}{\bigl|(d_j u'_j, v'_j)^\intercal\bigr|}(d_j u'_j, v'_j)^\intercal \in L, \quad v_j = \frac{1}{\bigl|(0, v'_j)^\intercal\bigr|}(0, v'_j)^\intercal \in L',$$

for $j = 1, \dots, i$. By construction, $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ are orthonormal bases of L and L' respectively. Moreover, for $au_j + bv_j \in \operatorname{lin}\{u_j, v_j\}$ and $cu_k + dv_k \in \operatorname{lin}\{u_k, v_k\}$ with $j \neq k$, $j, k \in \{1, \dots, i\}$, we get

$$\langle au_j + bv_j, cu_k + dv_k \rangle = ad \,\langle u_j, v_k \rangle + bc \,\langle v_j, u_k \rangle = 0,$$

i.e., the 2-dimensional linear subspaces $\operatorname{lin}\{u_1, v_1\}, \dots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal, as required.

Case (ii): Now we assume $L \cap L'^{\perp} \neq \{0\}$. Since $L^{\perp} \cap L' = (L + L'^{\perp})^{\perp}$, we have

$$\begin{aligned}\dim(L^{\perp} \cap L') &= \dim(L + L'^{\perp})^{\perp} = 2i - \dim L - \dim L'^{\perp} + \dim(L \cap L'^{\perp}) \\ &= \dim(L \cap L'^{\perp}).\end{aligned}$$

So, let $m = \dim(L^{\perp} \cap L') = \dim(L \cap L'^{\perp})$, $0 < m \leq i$, and let $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_m\}$ be orthonormal bases of $L \cap L'^{\perp}$ and $L^{\perp} \cap L'$, respectively. We define $\bar{L} = (L \cap L'^{\perp}) + (L^{\perp} \cap L')$. Then

$$\bar{L}^{\perp} \cap L = \bigl[(L^{\perp} + L') \cap (L + L'^{\perp})\bigr] \cap L = (L^{\perp} + L') \cap L$$

and hence

$$\begin{aligned}\dim(\bar{L}^{\perp} \cap L) &= \dim\bigl((L^{\perp} + L') \cap L\bigr) \\ &= \bigl[\dim L^{\perp} + \dim L' - \dim(L^{\perp} \cap L')\bigr] + \dim L - \dim(L^{\perp} + L' + L) \\ &= i + i - m + i - 2i = i - m.\end{aligned}$$

Analogously we get $\dim(\bar{L}^{\perp} \cap L') = i - m$. Moreover it is clear that the intersection $(\bar{L}^{\perp} \cap L) \cap (\bar{L}^{\perp} \cap L')^{\perp} = \{0\}$, and thus we can apply the previous case (i) to the subspaces $\bar{L}^{\perp} \cap L, \bar{L}^{\perp} \cap L' \subset \bar{L}^{\perp}$ to get orthonormal bases $\{u_{m+1}, \dots, u_i\}$ and $\{v_{m+1}, \dots, v_i\}$ of $\bar{L}^{\perp} \cap L$ and $\bar{L}^{\perp} \cap L'$ respectively, such that the 2-dimensional subspaces $\operatorname{lin}\{u_{m+1}, v_{m+1}\}, \dots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal. Embedding these vectors in the canonical way in $\mathbb{R}^{2i}$ we get orthonormal bases of L and L' verifying the required property. $\square$

Lemma 2.2. *Let $L, L' \in \mathcal{L}^n_i$. There exist orthonormal bases $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ of L and L' respectively, such that $\langle u_j, v_j \rangle \geq 0$ for all $j = 1, \dots, i$ and such that the vectors $\{u_1 + v_1, \dots, u_i + v_i\}$ are pairwise orthogonal.*

Proof. Let $k = \dim L \cap L' \leq i$ and let $w_1, \dots, w_k$ be an orthonormal basis of $L \cap L'$. Then we define $u_j = w_j \in L$ and $v_j = w_j \in L'$, for all $1 \leq j \leq k$. The vectors $\{u_1 + v_1, \dots, u_k + v_k\}$ are trivially pairwise orthogonal since $u_j + v_j = 2w_j$, and moreover, $\langle u_j, v_j \rangle = 1$, $j = 1, \dots, k$. So, they verify the required properties, and we have to complete them to bases of L and L'.

Let $\bar{L} = L \cap L'$ and consider $L \cap \bar{L}^{\perp}$ and $L' \cap \bar{L}^{\perp}$. Notice that

$$\dim L \cap \bar{L}^{\perp} = \dim L + \dim \bar{L}^{\perp} - \dim(L + \bar{L}^{\perp}) = i + (n - k) - n = i - k,$$

since $L + \bar{L}^{\perp} = \mathbb{R}^n$. Analogously $\dim L' \cap \bar{L}^{\perp} = i - k$. Moreover,

$$\big(L \cap \bar{L}^{\perp}\big) \cap \big(L' \cap \bar{L}^{\perp}\big) = L \cap L' \cap \bar{L}^{\perp} = \bar{L} \cap \bar{L}^{\perp} = \{0\},$$

and thus we can apply Lemma 2.1 to the subspaces $L \cap \bar{L}^{\perp}, L' \cap \bar{L}^{\perp} \in \mathcal{L}^n_{i-k}$ to get the existence of orthonormal bases

$$\{u_{k+1}, \dots, u_i\} \subset L \cap \bar{L}^{\perp} \quad \text{and} \quad \{v_{k+1}, \dots, v_i\} \subset L' \cap \bar{L}^{\perp}$$

such that the subspaces $\operatorname{lin}\{u_{k+1}, v_{k+1}\}, \dots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal. Notice that the vectors v_j can be chosen such that $\langle u_j, v_j \rangle \geq 0$ for all $j = k + 1, \dots, i$, otherwise we just have to replace v_j by $-v_j$. Since $u_j, v_j \in \bar{L}^{\perp}$ for all $j = k + 1, \dots, i$, together with the previously selected vectors, we obtain orthonormal bases $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ of L and L' respectively, verifying also that $\langle u_j, v_j \rangle \geq 0$ for all $j = 1, \dots, i$. Moreover, since $u_j + v_j \in \operatorname{lin}\{u_j, v_j\}$ for $j = k + 1, \dots, i$ and these 2-dimensional subspaces are pairwise orthogonal, we also get the required orthogonality property for the vectors $u_j + v_j$, $j = 1, \dots, i$. $\square$

3. Proofs of the main results

We start by proving bounds for the outer radii $\mathrm{R}_i(K+K')$ of the Minkowski sum of convex bodies in terms of the sum of the radii.

Proof of Theorem 1.1. The lower bound for $\mathrm{R}_1(K + K')$ (minimal width) is well-known (cf. (2)), and equality holds for instance when $K = K' = B_n$. So we prove (3) for $i = 2, \dots, n$.

Let $L \in \mathcal{L}^n_i$. Without loss of generality we may assume that $\mathrm{R}(K|L)B_{i,L}$ and $\mathrm{R}(K'|L)B_{i,L}$ are the circumballs of $K|L$ and $K'|L$ respectively. Then it is well-known (see [4, p. 59]) that there exist contact points

$$\begin{aligned} \{u_1, \dots, u_k\} &\subseteq \operatorname{relbd}(K|L) \cap \operatorname{relbd}\big(\mathrm{R}(K|L)B_{i,L}\big), \\ \{v_1, \dots, v_l\} &\subseteq \operatorname{relbd}(K'|L) \cap \operatorname{relbd}\big(\mathrm{R}(K'|L)B_{i,L}\big), \end{aligned}$$

with $2 \leq k, l \leq i + 1$, such that $0 \in \operatorname{conv}\{u_1, \dots, u_k\} \cap \operatorname{conv}\{v_1, \dots, v_l\}$.

Now we assume that there exist $t \in L$ and $\rho < \big(\mathrm{R}(K|L)^2 + \mathrm{R}(K'|L)^2\big)^{1/2}$ such that $(K + K')|L \subseteq t + \rho B_{i,L}$, and we will get a contradiction.

Notice first that since $0 \in \operatorname{conv}\{u_1, \dots, u_k\}$, there exists a point, say u_1, such that $\langle u_1, t \rangle \leq 0$: indeed, if for all $i = 1, \dots, k$ it holds $\langle u_i, t \rangle > 0$, then $\operatorname{conv}\{u_1, \dots, u_k\}$ and the origin 0 can be strictly separated by a hyperplane with (outer) normal vector t (see [15, p. 12]), which contradicts the fact that $0 \in \operatorname{conv}\{u_1, \dots, u_k\}$. Then we get

$$|u_1 - t|^2 = \mathrm{R}(K|L)^2 - 2\langle u_1, t\rangle + |t|^2 \geq \mathrm{R}(K|L)^2.$$

Next we take the vector $u_1 - t$. Notice that $u_1 - t \neq 0$ because $\langle u_1, t\rangle \leq 0$ and $u_1 \neq 0$. Since $0 \in \operatorname{conv}\{v_1, \dots, v_l\}$, an analogous argument to the previous one shows that there exists a point, say v_1, such that $\langle v_1, u_1 - t\rangle \geq 0$. Finally we consider the point

$$u_1 + v_1 \in K|L + K'|L = (K + K')|L \subseteq t + \rho B_{i,L},$$

for which, using the above conditions, we get

$$|u_1 + v_1 - t|^2 = |u_1 - t|^2 + 2\langle u_1 - t, v_1\rangle + |v_1|^2 \geq \mathrm{R}(K|L)^2 + \mathrm{R}(K'|L)^2 > \rho^2,$$

a contradiction. Therefore $\rho \geq \bigl(\mathrm{R}(K|L)^2 + \mathrm{R}(K'|L)^2\bigr)^{1/2}$ and, in particular, the same holds for the circumradius of $(K + K')|L$. Hence we finally get

$$\mathrm{R}\bigl((K + K')|L\bigr) \geq \bigl(\mathrm{R}(K|L)^2 + \mathrm{R}(K'|L)^2\bigr)^{1/2} \geq \frac{\sqrt{2}}{2}\bigl(\mathrm{R}(K|L) + \mathrm{R}(K'|L)\bigr)$$

for all $L \in \mathcal{L}^n_i$. Now let $L_i \in \mathcal{L}^n_i$ be such that $\mathrm{R}_i(K+K') = \mathrm{R}\bigl((K+K')|L_i\bigr)$. Then we can conclude that

$$\begin{aligned}\mathrm{R}_i(K + K') = \mathrm{R}\bigl((K + K')|L_i\bigr) &\geq \frac{1}{\sqrt{2}}\bigl(\mathrm{R}(K|L_i) + \mathrm{R}(K'|L_i)\bigr)\\ &\geq \frac{1}{\sqrt{2}}\bigl(\mathrm{R}_i(K) + \mathrm{R}_i(K')\bigr),\end{aligned}$$

which proves (3) for $i = 2, \dots, n$.

It remains to be shown that these inequalities are best possible. We fix $i \in \{2, \dots, n\}$ and consider the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k], \quad K' = [-\mathrm{e}_2, \mathrm{e}_2] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k].$$

Here for $i = n$ we are just taking $K = [-\mathrm{e}_1, \mathrm{e}_1]$, $K' = [-\mathrm{e}_2, \mathrm{e}_2]$. Since K and K' are both $(n - i + 1)$-cubes with edges parallel to the coordinate axes and length 2, it is clear that $\mathrm{R}(K|L), \mathrm{R}(K'|L) \geq 1$ for all $L \in \mathcal{L}^n_i$. Moreover, if $L = \operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}$ then $\mathrm{R}(K|L) = \mathrm{R}(K'|L) = 1$. This shows that $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 1$. Now we take the sum

$$K + K' = [-\mathrm{e}_1, \mathrm{e}_1] + [-\mathrm{e}_2, \mathrm{e}_2] + 2\sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k],$$

an $(n - i + 2)$-dimensional parallelepiped with edges again parallel to the coordinate axes and lengths 2 and 4. Then it is easy to see that

$$\mathrm{R}_i(K + K') = \mathrm{R}\bigl((K + K')|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\bigr) = \sqrt{2} = \frac{1}{\sqrt{2}}\bigl(\mathrm{R}_i(K) + \mathrm{R}_i(K')\bigr),$$

which concludes the proof of the theorem. $\square$

We already know that there exists also an upper bound for $\mathrm{R}_n(K+K')$ in terms of the sum of the circumradii, namely, $\mathrm{R}_n(K+K')\leq\mathrm{R}_n(K)+\mathrm{R}_n(K')$ (cf. (2)). To see that it is best possible, take $K=K'=B_n$. However:

Remark 3.1. *For any $i\in\{1,\dots,n-1\}$ fixed, define the convex bodies*

$$K=[-\mathrm{e}_{n-i+1},\mathrm{e}_{n-i+1}]\quad\text{and}\quad K'=\sum_{k=1}^{n-i}[-\mathrm{e}_k,\mathrm{e}_k].$$

Notice that $K|\operatorname{lin}\{\mathrm{e}_{n-i},\mathrm{e}_{n-i+2},\dots,\mathrm{e}_n\}=K'|\operatorname{lin}\{\mathrm{e}_{n-i+1},\dots,\mathrm{e}_n\}=\{0\}$, and hence both $\mathrm{R}_i(K)=\mathrm{R}_i(K')=0$, i.e., $\mathrm{R}_i(K)+\mathrm{R}_i(K')=0$. However, $K+K'=\sum_{k=1}^{n-i+1}[-\mathrm{e}_k,\mathrm{e}_k]$ is an $(n-i+1)$-dimensional convex body, which implies that the dimension $\dim\big((K+K')|L\big)\geq 1$ for all $L\in\mathcal{L}_i^n$, and thus $\mathrm{R}(K+K')>0$. Hence we conclude that there exists no constant $c>0$ such that $c\,\mathrm{R}_i(K+K')\leq\mathrm{R}_i(K)+\mathrm{R}_i(K')$ for any $i=1,\dots,n-1$.

Now we get the corresponding bounds for the inner radii $\mathrm{r}_i(K+K')$ by proving Theorem 1.2.

Proof of Theorem 1.2. The lower bound for $\mathrm{r}_n(K+K')$ (inradius) is well-known (cf. (2)), and equality holds for instance when $K=K'=B_n$. So we prove (4) for $i=1,\dots,n-1$.

Without loss of generality we may assume that $\mathrm{r}_i(K)=\mathrm{r}(K\cap L;L)$ and $\mathrm{r}_i(K')=\mathrm{r}(K'\cap L';L')$ for $L,L'\in\mathcal{L}_i^n$, i.e., that the greatest i-dimensional balls contained in K and K' are $\mathrm{r}(K\cap L;L)B_{i,L}$ and $\mathrm{r}(K'\cap L';L')B_{i,L'}$, respectively. For the sake of brevity we write $\mathrm{r}=\mathrm{r}(K\cap L;L)=\mathrm{r}_i(K)$ and $\mathrm{r}'=\mathrm{r}(K'\cap L';L')=\mathrm{r}_i(K')$. Thus it suffices to show that inequality (4) holds for i-dimensional balls, i.e., that

$$\sqrt{2}\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq\mathrm{r}+\mathrm{r}',\tag{5}$$

since, taking into account that $\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}\subseteq K+K'$, we have

$$\sqrt{2}\mathrm{r}_i(K+K')\geq\sqrt{2}\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq\mathrm{r}+\mathrm{r}'=\mathrm{r}_i(K)+\mathrm{r}_i(K').$$

So we have to prove (5). By Lemma 2.2 we can assure the existence of two subsets of pairwise orthogonal vectors

$$\{u_1,\dots,u_i\}\in\operatorname{bd}(\mathrm{r}B_{i,L})\quad\text{and}\quad\{v_1,\dots,v_i\}\in\operatorname{bd}(\mathrm{r}'B_{i,L'}),$$

such that $\{u_1+v_1,\dots,u_i+v_i\}$ are also pairwise orthogonal with $\langle u_j,v_j\rangle\geq 0$, $j=1,\dots,i$. Let $\bar{L}=\operatorname{lin}\{u_1+v_1,\dots,u_i+v_i\}\in\mathcal{L}_i^n$. Next we show that the i-dimensional ball

$$\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\subset\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}.\tag{6}$$

Notice first that

$$|u_j+v_j|^2=|u_j|^2+|v_j|^2+2\langle u_j,v_j\rangle\geq\mathrm{r}^2+(\mathrm{r}')^2.$$

Then, denoting by $\mathcal{E}=\left\{\sum_{j=1}^i\lambda_j(u_j+v_j):\lambda_j\in[-1,1],\sum_{j=1}^i\lambda_j^2\leq 1\right\}$ the 0-symmetric ellipsoid with semi-axes $\{u_j+v_j,j=1,\dots,i\}$, it trivially holds that $\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\subseteq\mathcal{E}$. Thus, in order to show (6) it suffices to prove the inclusion $\mathcal{E}\subset \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}$, i.e., that $\sum_{j=1}^i\lambda_j(u_j+v_j)\in \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}$ for $\sum_{j=1}^i\lambda_j^2=1$. Clearly, $\sum_{j=1}^i\lambda_ju_j\in L$, and moreover, since $\{u_1,\dots,u_i\}$ are pairwise orthogonal vectors with $|u_j|=\mathrm{r}$, we have

$$\left|\sum_{j=1}^i\lambda_ju_j\right|^2=\sum_{j=1}^i\lambda_j^2\,|u_j|^2=\mathrm{r}^2\sum_{j=1}^i\lambda_j^2=\mathrm{r}^2.$$

Therefore, $\sum_{j=1}^i\lambda_ju_j\in \mathrm{r}B_{i,L}$. Analogously we get $\sum_{j=1}^i\lambda_jv_j\in \mathrm{r}'B_{i,L'}$ and thus $\sum_{j=1}^i\lambda_j(u_j+v_j)=\sum_{j=1}^i\lambda_ju_j+\sum_{j=1}^i\lambda_jv_j\in \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}$. This shows (6) and we can conclude that

$$\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq \mathrm{r}_i\left(\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\right)=\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}\geq\frac{1}{\sqrt{2}}(\mathrm{r}+\mathrm{r}'),$$

which gives the required inequality (5).

It remains to be shown that these inequalities are best possible. We fix $i\in\{1,\dots,n-1\}$. Let $j=2i-n$ if $2i\geq n$, and $j=0$ otherwise, and consider the i-dimensional linear subspaces

$$L=\operatorname{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_j,\mathrm{e}_{j+1},\dots,\mathrm{e}_i\},\quad L'=\operatorname{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_j,\mathrm{e}_{i+1},\dots,\mathrm{e}_{2i-j}\}.$$

We are going to show that equality in (4) is attained for the i-dimensional unit balls $B_{i,L}$ and $B_{i,L'}$. Notice that if we prove the inequality

$$\mathrm{r}_i(B_{i,L}+B_{i,L'})\leq\sqrt{2}\tag{7}$$

then by (4) we can conclude that

$$\sqrt{2}\geq \mathrm{r}_i(B_{i,L}+B_{i,L'})\geq\frac{1}{\sqrt{2}}\left[\mathrm{r}_i(B_{i,L})+\mathrm{r}_i(B_{i,L'})\right]=\frac{1}{\sqrt{2}}\,(1+1)=\sqrt{2},$$

which gives the required result. Observe first that since $B_{i,L}+B_{i,L'}$ is a 0-symmetric convex body, for any $\bar{L}\in\mathcal{L}_i^n$ we have

$$\max_{x\in\bar{L}^\perp}\mathrm{r}\big((B_{i,L}+B_{i,L'})\cap(x+\bar{L});x+\bar{L}\big)=\mathrm{r}\big((B_{i,L}+B_{i,L'})\cap\bar{L};\bar{L}\big).$$

Therefore in order to show (7) it suffices to prove that

$$\mathrm{r}\big((B_{i,L}+B_{i,L'})\cap\bar{L};\bar{L}\big)\leq\sqrt{2}\quad\text{for all }\ \bar{L}\in\mathcal{L}_i^n.\tag{8}$$

If $\dim\big((B_{i,L}+B_{i,L'})\cap\bar{L}\big)<i$ for $\bar{L}\in\mathcal{L}_i^n$ then $\mathrm{r}\big((B_{i,L}+B_{i,L'})\cap\bar{L};\bar{L}\big)=0$. So we take $\bar{L}\in\mathcal{L}_i^n$ with $\dim\big((B_{i,L}+B_{i,L'})\cap\bar{L}\big)=i$. Notice that if we find $x\in\operatorname{relbd}\big((B_{i,L}+B_{i,L'})\cap\bar{L}\big)$ with $|x|\leq\sqrt{2}$, then we immediately get (8). In order to find such an x, let $L''=\operatorname{lin}\{\mathrm{e}_{j+1},\dots,\mathrm{e}_n\}$. If $j=2i-n$ (i.e., if $2i\geq n$) then

$$\begin{aligned}\dim(\bar{L}\cap L'')&=\dim\bar{L}+\dim L''-\dim(\bar{L}+L'')=i+n-j-\dim(\bar{L}+L'')\\&\geq i+n-j-n=i-j=i-2i+n=n-i\geq 1,\end{aligned}$$

and moreover, $L+L'=\mathbb{R}^n$, i.e., $\dim(B_{i,L}+B_{i,L'})=n$. On the other hand, if $j=0$ then $L''=\mathbb{R}^n$, and so $\bar{L}\cap L''=\bar{L}$. Therefore, in both cases,

$\dim\bigl((B_{i,L}+B_{i,L'})\cap\bar{L}\cap L''\bigr)\geq 1$, which ensures the existence of a boundary point $x\in\operatorname{relbd}(B_{i,L}+B_{i,L'})\cap\bar{L}\cap L''$. Since any $x\in\operatorname{relbd}(B_{i,L}+B_{i,L'})\cap L''$ is expressed in the form

$$x=\sum_{k=j+1}^{i}\lambda_k\mathrm{e}_k+\sum_{k=i+1}^{2i-j}\mu_k\mathrm{e}_k,\quad\text{ with }\sum_{k=j+1}^{i}\lambda_k^2=1,\ \sum_{k=i+1}^{2i-j}\mu_k^2=1,$$

we trivially get

$$|x|^2=\sum_{k=j+1}^{i}\lambda_k^2+\sum_{k=i+1}^{2i-j}\mu_k^2=2.$$

This shows (8) and concludes the proof. □

We already know that there exists also an upper bound for $\mathrm{r}_1(K+K')$ in terms of the sum of the diameters, namely, $\mathrm{r}_1(K+K')\leq\mathrm{r}_1(K)+\mathrm{r}_1(K')$ (cf. (2)). It is best possible, as shown by just taking $K=K'=B_n$. However:

Remark 3.2. *For any $i\in\{2,\dots,n\}$ fixed, define the convex bodies*

$$K=[-\mathrm{e}_1,\mathrm{e}_1]\quad\text{ and }\quad K'=\sum_{k=2}^{i}[-\mathrm{e}_k,\mathrm{e}_k].$$

Since K and K' are, respectively, 1-dimensional and $(i-1)$-dimensional convex bodies, $\mathrm{r}_i(K)=\mathrm{r}_i(K')=0$. However, $K+K'=\sum_{k=1}^{i}[-\mathrm{e}_k,\mathrm{e}_k]$ and clearly $\mathrm{r}_i(K+K')=1$. Hence we can conclude that there exists no constant $c>0$ such that $c\,\mathrm{r}_i(K+K')\leq\mathrm{r}_i(K)+\mathrm{r}_i(K')$ for any $i=2,\dots,n$.

The bounds obtained in Theorems 1.1 and 1.2 can be improved when special sums of convex bodies are considered. Moreover, reverse inequalities to (3) and (4) exist for these special sets (cf. Remarks 3.1 and 3.2). We deal with this question in the last section.

4. Special sums of convex bodies

Observe that equality in (3) and (4) is attained in both cases for convex bodies with empty interior. Also the non-existence of the reverse inequalities is due to this particular type of bodies (cf. Remarks 3.1 and 3.2). Thus the question arises whether those inequalities can be improved if convex bodies with non-empty interior are considered. So we ask, in particular, for the special case when one of the bodies involved is the Euclidean ball.

Proposition 4.1. *Let $K\in\mathcal{K}^n$ and $r\geq 0$. Then for all $i=1,\dots,n$,*

$$\mathrm{R}_i(K+rB_n)=\mathrm{R}_i(K)+r\quad\text{ and }\quad\mathrm{r}_i(K+rB_n)\geq\mathrm{r}_i(K)+r.$$

All inequalities are best possible and for $i=2,\dots,n-1$ they can be strict.

Proof. The identity for R_i is a straightforward computation:

$$\mathrm{R}_i(K+rB_n)=\min_{L\in\mathcal{L}_i^n}\mathrm{R}\bigl((K+rB_n)|L\bigr)=\min_{L\in\mathcal{L}_i^n}\mathrm{R}(K|L+rB_n|L)=\mathrm{R}_i(K)+r.$$

Now we show the lower bound for $\mathrm{r}_i(K+rB_n)$. First notice that for any $L\in\mathcal{L}_i^n$ and $x\in\mathbb{R}^n$, we have $K\cap(x+L)+rB_{i,L}\subseteq(K+rB_n)\cap(x+L)$. Indeed, if $z\in K\cap(x+L)+rB_{i,L}$ then $z=x+l+ru$, where $l\in L$, $x+l\in K$ and $u\in B_{i,L}$, and thus $z=x+l+ru\in(K+rB_n)\cap(x+L)$.

Let $L_i\in\mathcal{L}_i^n$ and $x\in L_i^\perp$ be such that $\mathrm{r}_i(K)=\mathrm{r}\bigl(K\cap(x+L_i);x+L_i\bigr)$. Then using the above property we get

$$\begin{aligned}\mathrm{r}_i(K+rB_n)&\geq\mathrm{r}\bigl((K+rB_n)\cap(x+L_i);x+L_i\bigr)\\&\geq\mathrm{r}\bigl(K\cap(x+L_i)+rB_{i,L_i};x+L_i\bigr)\\&=\mathrm{r}\bigl(K\cap(x+L_i);x+L_i\bigr)+r=\mathrm{r}_i(K)+r.\end{aligned}$$

Equality holds, for instance, if $K=B_n$. Finally we show that, unlike the R_i case, there exist convex bodies with $\mathrm{r}_i(K+rB_n)>\mathrm{r}_i(K)+r$.

Let $P_\varepsilon=\mathrm{conv}\{\pm p_1,\pm p_2,\pm p_3\}$ be the non-regular triangular antiprism in $\mathbb{R}^3$ with vertices $p_1=\bigl(1/\sqrt{3},1,\varepsilon\bigr)$, $p_2=\bigl(1/\sqrt{3},-1,\varepsilon\bigr)$, $p_3=\bigl(-2/\sqrt{3},0,\varepsilon\bigr)$, $\varepsilon>0$ (see Figure 1). First we prove that $\mathrm{r}_2(P_\varepsilon)=\sqrt{3}/2$ for ε small enough.

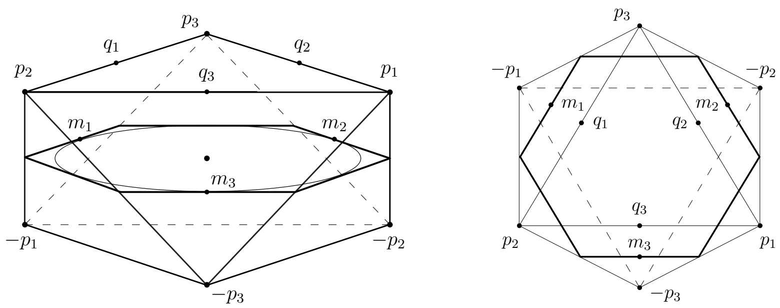

FIGURE 1. Triangular antiprism with $\mathrm{r}_2(P_\varepsilon+rB_3)>\mathrm{r}_2(P_\varepsilon)+r$.

Let $q_1=(1/2)(p_2+p_3)$, $q_2=(1/2)(p_1+p_3)$, $q_3=(1/2)(p_1+p_2)$ be the middle points of the edges of the triangle contained in the plane $z=\varepsilon$, and let $m_j=(1/2)(-p_j+q_j)$, $j=1,2,3$ (see Figure 1). It is easy to check that $|m_j|=\sqrt{3}/2$ and $|q_j|=\sqrt{1/3+\varepsilon^2}$, for all $j=1,2,3$. Then $|q_j|\leq\sqrt{3}/2$ if and only if $\varepsilon\leq\sqrt{5/12}$ and hence, for any $\varepsilon\leq\sqrt{5/12}$, all segments

$$\Bigl\{\bigl[(0,0,\varepsilon),q_j\bigr],[q_j,m_j]:j=1,2,3\Bigr\}\subset\mathrm{bd}\,P_\varepsilon\cap\frac{\sqrt{3}}{2}B_3.$$

Now we can prove that $\mathrm{r}_2(P_\varepsilon)=\sqrt{3}/2$ for $\varepsilon\leq\sqrt{5/12}$. Notice that since P_ε is 0-symmetric, $\mathrm{r}_2(P_\varepsilon)=\max_{L\in\mathcal{L}_2^3}\mathrm{r}\bigl(P_\varepsilon\cap L;L\bigr)$. If $L=\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$, then $P_\varepsilon\cap\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$ is the regular hexagon with apothem $|m_j|$, and so with incircle $(\sqrt{3}/2)B_{2,\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}$ (see Figure 1). Therefore, $\mathrm{r}(P_\varepsilon\cap\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\};\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\})=\sqrt{3}/2$. Now let $L\in\mathcal{L}_2^3$, $L\neq\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$. Clearly $L\cap\mathrm{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$ is a 1-dimensional subspace which intersects the relative interior of, at least, one of the segments with end-points m_j, $j=1,2,3$, say $[m_1,m_2]$. Then there is a point

$$q\in L\cap\operatorname{bd}P_\varepsilon\cap\Big\{[m_1,q_1],\big[q_1,(0,0,\varepsilon)\big],\big[(0,0,\varepsilon),q_2\big],[q_2,m_2]\Big\}$$

with $|q|\leq\sqrt{3}/2$, which ensures that $\mathrm{r}(P_\varepsilon\cap L;L)\leq\sqrt{3}/2$ for all $L\in\mathcal{L}^3_2$, $L\neq\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$. Thus we can conclude that $\mathrm{r}_2(P_\varepsilon)=\sqrt{3}/2$ if $\varepsilon\leq\sqrt{5/12}$. Finally, if we show that

$$\mathrm{r}_2(P_\varepsilon+rB_3)\geq 1+\sqrt{r^2-\varepsilon^2}\qquad\text{for } r\geq\varepsilon, \tag{9}$$

then we will conclude that

$$\mathrm{r}_2(P_\varepsilon+rB_3)\geq 1+\sqrt{r^2-\varepsilon^2}>\frac{\sqrt{3}}{2}+r=\mathrm{r}_2(P_\varepsilon)+r$$

for $\varepsilon\leq\sqrt{5/12}$ and $r>\big(2+\sqrt{3}\big)\big(\varepsilon^2-\sqrt{3}+7/4\big)\geq\varepsilon$, as required.

Observe that in order to prove (9) it suffices to show that

$$\mathrm{r}\big((P_\varepsilon+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\};\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\big)\geq 1+\sqrt{r^2-\varepsilon^2}.$$

Denoting by $\pm\bar{p}_j=\pm p_j|\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$, it is a straightforward computation to check that $(\pm p_j+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}=\pm\bar{p}_j+\sqrt{r^2-\varepsilon^2}B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}$. Since $(\pm p_j+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\subset(P_\varepsilon+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$,

$$\begin{aligned}(P_\varepsilon+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}&\supset\operatorname{conv}\Big\{\pm\bar{p}_j+\sqrt{r^2-\varepsilon^2}B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}:j=1,2,3\Big\}\\&=\big(P_\varepsilon|\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\big)+\sqrt{r^2-\varepsilon^2}B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}.\end{aligned}$$

Notice that the projected body $H=P_\varepsilon|\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$ is the regular hexagon in the plane $\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}$ with vertices $\pm\bar{p}_j$, $j=1,2,3$, which has 2-dimensional inradius $\mathrm{r}\big(H;\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\big)=1$. Thus,

$$\begin{aligned}&\mathrm{r}\big((P_\varepsilon+rB_3)\cap\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\};\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\big)\\&\qquad\geq\mathrm{r}\Big(H+\sqrt{r^2-\varepsilon^2}B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}};\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\Big)\\&\qquad=\mathrm{r}\big(H;\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}\big)+\sqrt{r^2-\varepsilon^2}=1+\sqrt{r^2-\varepsilon^2},\end{aligned}$$

which shows (9) and finishes the proof. □

Remark 4.1. *There exist upper bounds for* $\mathrm{r}_i(K+rB_n)$ *in terms of* $\mathrm{r}_i(K)$ *and* r. *Namely, using* (1) *and Proposition 4.1, it is straightforward to get*

$$\mathrm{r}_i(K+rB_n)\leq\mathrm{R}_{n-i+1}(K+rB_n)=\mathrm{R}_{n-i+1}(K)+r<(i+1)\mathrm{r}_i(K)+r,$$

although this bound is far from being optimal.

Remark 4.2. *Let* $K,K'\in\mathcal{K}^n$. *If* K' *has non-empty interior then we have* $\mathrm{r}_i(K+K')\geq\mathrm{r}_i\big(K+\mathrm{r}(K')B_n\big)\geq\mathrm{r}_i(K)+\mathrm{r}(K')$. *Thus in order to improve the constant* $\sqrt{2}$ *in* (4) *the inradius of the body has to be involved.*

Remark 4.3. *Notice that the family of triangular antiprisms P_ε considered in the proof of Proposition 4.1 shows also that the functional* $\mathrm{r}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$, $i = 2, \dots, n-1$, *is not continuous with respect to the Hausdorff metric: using the previous notation and taking* $\varepsilon = 1/k$, *we have* $\lim_{k\to\infty} P_{1/k} = H$ *but*

$$\lim_{k\to\infty} \mathrm{r}_2(P_{1/k}) = \frac{\sqrt{3}}{2} < 1 = \mathrm{r}_2(H).$$

However, it is easy to see that $\mathrm{r}_i : \{K \in \mathcal{K}^n : \dim K = n\} \longrightarrow \mathbb{R}_{\geq 0}$ *is a continuous map.*

The *central symmetral* of $K \in \mathcal{K}^n$ is defined as $K^0 = \bigl(K + (-K)\bigr)/2 = (K - K)/2$ (see [4, p. 79]). We are interested in the behavior of the radii regarding the special case of the Minkowski sum $K - K$. In [10, Lemma 2.1, Remark 2.1] it was shown that $\mathrm{R}_i(K^0) \leq \mathrm{R}_i(K)$ and $\mathrm{r}_i(K^0) \geq \mathrm{r}_i(K)$ for all $i = 1, \dots, n$. The next proposition completes this particular case, by showing that the bounds in (3) and (4) can be improved and that there are non-trivial reverse inequalities (cf. Remarks 3.1 and 3.2).

Proposition 4.2. *Let* $K \in \mathcal{K}^n$. *Then for all* $i = 1, \dots, n$,

$$\begin{aligned} &a) \quad \sqrt{2}\sqrt{\frac{i+1}{i}}\mathrm{R}_i(K) \leq \mathrm{R}_i(K-K) \leq 2\mathrm{R}_i(K), \\ &b) \quad\quad\quad\quad 2\mathrm{r}_i(K) \leq \mathrm{r}_i(K-K) < 2(i+1)\mathrm{r}_i(K). \end{aligned} \tag{10}$$

All inequalities except for the upper bound in (b) are best possible.

Proof. The right hand side in (10.a) and the left hand side in (10.b) are known (see [10, Lemma 2.1, Remark 2.1]). In order to prove the left inequality in (10.a) let $L_i \in \mathcal{L}_i^n$ be such that $\mathrm{R}_i(K-K) = \mathrm{R}\bigl((K-K)|L_i\bigr)$ for any fixed $i \in \{1, \dots, n\}$. It is clear that $K^0|L_i = (K|L_i)^0$. Then, since central symmetry preserves the diameter (see e.g. [4, p. 79]) and using the well-known Jung inequality (see e.g. [4, p. 84]) in dimension i, we get

$$\begin{aligned} \mathrm{R}_i(K-K) &= \mathrm{R}\bigl((K-K)|L_i\bigr) = 2\mathrm{R}\bigl(K^0|L_i\bigr) = 2\mathrm{R}\bigl((K|L_i)^0\bigr) = \mathrm{D}\bigl((K|L_i)^0\bigr) \\ &= \mathrm{D}(K|L_i) \geq \sqrt{\frac{2(i+1)}{i}}\mathrm{R}(K|L_i) \geq \sqrt{\frac{2(i+1)}{i}}\mathrm{R}_i(K). \end{aligned}$$

Equality in the Jung inequality holds for the i-dimensional regular simplex S_i as well as for every convex body of diameter D containing the regular simplex of edge-length D. Hence, in our case, equality holds for any convex body K such that $\mathrm{R}_i(K) = \mathrm{R}(K|L_i)$ and such that $K|L_i$ is an extremal set in Jung's inequality. For instance, equality holds for $K = S_i + MC_{n-i}$, where $C_{n-i} \subset (\operatorname{aff} S_i)^\perp$ represents the $(n-i)$-dimensional unit cube and $M > 0$ is sufficiently large.

The right hand side in (10.b) is a direct consequence of (1) and the already mentioned property of the central symmetrization, $\mathrm{R}_i(K^0) \leq \mathrm{R}_i(K)$:

$$\mathrm{r}_i(K-K) = 2\mathrm{r}_i(K^0) \leq 2\mathrm{R}_{n-i+1}(K^0) \leq 2\mathrm{R}_{n-i+1}(K) < 2(i+1)\mathrm{r}_i(K). \quad \square$$

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